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The modified Mann type iterative algorithm for a countable family of totally quasi-ϕ-asymptotically nonexpansive mappings by the hybrid generalized f-projection method

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Abstract
The purpose of this article is to introduce the modified Mann type iterative sequence, using a new technique, by the hybrid generalized f-projection operator for a countable family of totally quasi-ϕ-asymptotically nonexpansive mappings in a uniform smooth and strictly convex Banach space with the Kadec-Klee property. Then we prove that the modified Mann type iterative scheme converges strongly to a common element of the sets of fixed points of the given mappings. Our result extends and improves the results of Li *et al.* (Comput. Math. Appl. 60:1322-1331, 2010), Takahashi *et al.* (J. Math. Anal. Appl. 341:276-286, 2008) and many other authors.
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1 Introduction

Let E be a real Banach space and C be a nonempty closed and convex subset of E. A mapping $T: C \to C$ is said to be *totally asymptotically nonexpansive* [1] if there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\left\|T^n x - T^n y\right\| \le \|x-y\| + \mu_n \psi\left(\|x-y\|\right) + \nu_n, \quad \forall x, y \in C, n \ge 1.$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the fixed point set of T, that is, $F(T) = \{x \in C : Tx = x\}$. A point $p \in C$ is called an *asymptotic fixed point* of T [2] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The asymptotic fixed point set of T is denoted by $\widehat{F}(T)$.

Let E^* be a dual space of the Banach space E. We recall that for all $x \in E$ and $x \in E^*$, we denote the value of x^* at x by $\langle x, x^*\rangle$. Then the *normalized duality mapping* $J: E \to 2^{E^*}$ is defined by

$$J(x) = \left\{x^* \in E^* : \left\langle x, x^*\right\rangle = \|x\|^2, \left\|x^*\right\| = \|x\|\right\}, \quad \forall x \in E.$$

If E is a Hilbert space, then $J = I$, where I is the identity mapping. Next, consider the functional $\phi : E \times E \to \mathbb{R}^+ \cup \{0\}$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E, \tag{1.1}$$

where J is the normalized duality mapping and $\langle\cdot,\cdot\rangle$ denotes the duality pairing of E and E^*.

If E is a Hilbert space, then $\phi(x,y) = \|x-y\|^2$. It is obvious from the definition of ϕ that

$$\big(\|x\| - \|y\|\big)^2 \le \phi(x,y) \le \big(\|x\| + \|y\|\big)^2, \quad \forall x,y \in E. \tag{1.2}$$

T is said to be *relatively nonexpansive* [3, 4] if $\widehat{F}(T) = F(T)$ and

$$\phi(p, Tx) \le \phi(p,x), \quad \forall x \in C, p \in F(T).$$

T is said to be *relatively asymptotically nonexpansive* [5, 6] if $\widehat{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi\big(p, T^n x\big) \le k_n \phi(p,x), \quad \forall x \in C, p \in F(T), n \ge 1.$$

T is said to be *ϕ-nonexpansive* [7, 8] if

$$\phi(Tx, Ty) \le \phi(x,y), \quad \forall x,y \in C.$$

T is said to be *quasi-ϕ-nonexpansive* [7, 8] if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p,x), \quad \forall x \in C, p \in F(T).$$

T is said to be *asymptotically ϕ-nonexpansive* [8] if there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi\big(T^n x, T^n y\big) \le k_n \phi(x,y), \quad \forall x,y \in C.$$

T is said to be *quasi-ϕ-asymptotically nonexpansive* [8] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi\big(p, T^n x\big) \le k_n \phi(p,x), \quad \forall x \in C, p \in F(T), n \ge 1.$$

T is said to be *totally quasi-ϕ-asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi\big(p, T^n x\big) \le \phi(p,x) + \nu_n \varphi\big(\phi(p,x)\big) + \mu_n, \quad \forall x \in C, p \in F(T), n \ge 1.$$

Remark 1.1 (1) Every relatively nonexpansive mapping implies a relatively quasi-nonexpansive mapping, a quasi-ϕ-nonexpansive mapping implies a quasi-ϕ-asymptotically nonexpansive mapping and a quasi-ϕ-asymptotically nonexpansive mapping implies a totally quasi-ϕ-asymptotically nonexpansive mapping, but the converses are not true.

(2) A relatively quasi-nonexpansive mapping is sometimes called hemi-relatively nonexpansive mapping. The class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings (see [4, 9–13]), which requires the strong restriction $F(T) = \widehat{F}(T)$.

(3) For other examples of relatively quasi-nonexpansive mappings such as the generalized projections and others, see [7, Examples 2.3 and 2.4].

On the other hand, Alber [14] introduced that the *generalized projection* $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is a solution of the minimization problem

$$\phi(\bar{x},x) = \inf_{y\in C} \phi(y,x). \tag{1.3}$$

In 2006, Wu and Huang [15] introduced a new generalized f-projection operator in Banach spaces. They extended the definition of generalized projection operators introduced by Abler [16] and proved the properties of the generalized f-projection operator.

Now, we recall the concept of the generalized f-projection operator. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined by

$$G(y,\varpi) = \|y\|^2 - 2\langle y,\varpi\rangle + \|\varpi\|^2 + 2\rho f(y), \tag{1.4}$$

where $y \in C$, $\varpi \in E^*$, ρ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. From the definition of G, Wu and Huang [15] studied the following properties:

(1) $G(y,\varpi)$ is convex and continuous with respect to ϖ when y is fixed;

(2) $G(y,\varpi)$ is convex and lower semicontinuous with respect to y when ϖ is fixed.

Definition 1.2 Let E be a real Banach space with the dual space E^* and C be a nonempty closed and convex subset of E. We say that $\pi_C^f : E^* \to 2^C$ is a *generalized f-projection operator* if

$$\pi_C^f \varpi = \left\{ u \in C : G(u,\varpi) = \inf_{y\in C} G(y,\varpi), \forall \varpi \in E^* \right\}.$$

In 1953, Mann [17] introduced the following iteration process, which is now well known as Mann's iteration:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 1, \tag{1.5}$$

where the initial guess element $x_1 \in C$ is arbitrary and $\{\alpha_n\}$ is a sequence in $[0,1]$. Mann's iteration has been extensively investigated for nonexpansive mappings and some mappings. In an infinite-dimensional Hilbert space, Mann's iteration can conclude *only weak convergence* (see [18, 19]). Bauschke and Combettes [20] introduced a modified Mann iteration method (1.5) in a Hilbert space and proved, under appropriate conditions, some strong convergence.

Recently, Takahashi *et al.* [21] studied the strong convergence theorem by the new hybrid method $\{x_n\}$ for a family of nonexpansive mappings in Hilbert spaces: $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_{n+1} = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \geq 1$ and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. They proved that if $\{T_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)}x_0$.

The ideas to generalize the process (1.5) from Hilbert spaces to Banach spaces have recently been made. Especially, Matsushita and Takahashi [11] proposed the following hybrid iteration method with the generalized projection for a relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}x_0, \quad \forall n \geq 0. \end{cases} \tag{1.7}$$

They proved that $\{x_n\}$ converges strongly to a point $\Pi_{F(T)}x_0$. Many authors studied methods for approximating fixed points of a countable family of (relatively quasi-) nonexpansive mappings (see [22–26]).

In 2008, Alber *et al.* [27] proved a new strong convergence result of the regularized successive approximation method for a total asymptotically nonexpansive mapping in a Hilbert spaces. In 2010, Li *et al.* [28] introduced the following hybrid iterative scheme $\{x_n\}$ for approximation fixed points of a relatively nonexpansive mapping using the generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$ and

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}}x_0, \quad \forall n \geq 0. \end{cases} \tag{1.8}$$

They proved strong convergence theorems for finding an element in the fixed point set of T.

One question is raised naturally as follows:

Are the results of Alber et al. [27], *Li et al.* [28] *and Takahashi et al.* [21] *true in the framework of strictly convex Banach spaces for totally quasi-ϕ-asymptotically nonexpansive mappings?*

Motivated and inspired by the works mentioned above, in this article we aim to introduce a new hybrid projection algorithm of the generalized f-projection operator for a countable family of totally quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. Our result extends and improves the results of Li *et al.* [28], Takahashi *et al.* [21] and many other authors.

2 Preliminaries

A Banach space E with the norm $\|\cdot\|$ is called *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be a unit sphere of E. A Banach space E is called *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also called *uniformly smooth* if the limit exists uniformly for all $x, y \in U$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| \leq t\right\}.$$

The *modulus of convexity* of E (see [29]) is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}.$$

In this paper, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Remark 2.1 The basic properties of E, E^*, J and J^{-1} are as follows (see [30]):

(1) If E is an arbitrary Banach space, then J is monotone and bounded;
(2) If E is strictly convex, then J is strictly monotone;
(3) If E is smooth, then J is single-valued and semi-continuous;
(4) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;
(5) If E is reflexive smooth and strictly convex, then the normalized duality mapping J is single-valued, one-to-one and onto;
(6) If E is a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-valued, bijective and is also the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$;
(7) If E is uniformly smooth, then E is smooth and reflexive;
(8) E is uniformly smooth if and only if E^* is uniformly convex;
(9) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous.

Remark 2.2 If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.1) we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [30–32] for more details).

Recall that a Banach space E has the *Kadec-Klee property* [30, 31, 33] if, for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

The *generalized projection* [14] from E into C is defined by

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, x).$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and the strict monotonicity of the mapping J (see, for example, [14, 30, 31, 34, 35]). If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ and Π_C becomes the metric projection $P_C : H \to C$. If C is a nonempty closed and convex subset of a Hilbert space H, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces.

We also need the following lemmas for the proof of our main results.

Let T be a nonlinear mapping, T is said to be *uniformly asymptotically regular* on C if

$$\lim_{n\to\infty} \Bigl(\sup_{x \in C} \| T^{n+1}x - T^n x \| \Bigr) = 0.$$

A mapping T from C into itself is said to be *closed* if, for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, we have $Tx_0 = y_0$.

Lemma 2.3 (Chang *et al.* [36]) *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $T : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with the sequences ν_n and μ_n of nonnegative real numbers with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(T)$ is a closed convex subset of C.*

Lemma 2.4 (Wu and Hung [15]) *Let E be a real reflexive Banach space with the dual space E^* and C be a nonempty closed and convex subset of E. The following statements hold*:

(1) *$\pi_C^f \varpi$ is a nonempty, closed and convex subset of C for all $\varpi \in E^*$*;

(2) *If E is smooth, then for all $\varpi \in E^*$, $x \in \pi_C^f \varpi$ if and only if*

$$\langle x - y, \varpi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;$$

(3) *If E is strictly convex and $f : C \to \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$, where $x \in C$), then $\pi_C^f \varpi$ is a single-valued mapping.*

In the following lemma, Fan *et al.* [37] showed that Lemma 2.1(iii) in [37] can be removed.

Lemma 2.5 (Fan *et al.* [37]) *Let E be a real reflexive Banach space with its dual space E^* and C be a nonempty closed and convex subset of E. If E is strictly convex, then $\pi_C^f \varpi$ is single-valued.*

Note that J is a single-valued mapping when E is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$, where $x \in E$. This substitution in (1.4) gives the following:

$$G(y, Jx) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 + 2\rho f(y). \tag{2.1}$$

Now, we consider the second generalized f-projection operator in Banach spaces (see [28]).

Definition 2.6 Let E be a real smooth Banach space and C be a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is the *generalized f-projection operator* if

$$\Pi_C^f x = \Bigl\{u \in C : G(u, Jx) = \inf_{y \in C} G(y, Jx), \forall x \in E\Bigr\}.$$

Lemma 2.7 (Deimling [38]) *Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \ge \langle x, x^*\rangle + \alpha, \quad \forall x \in E.$$

Lemma 2.8 (Li *et al.* [28]) *Let E be a reflexive smooth Banach space and C be a nonempty, closed and convex subset of E. The following statements hold*:

(1) $\Pi_C^f x$ *is nonempty, closed and convex subset of C for all $x \in E$*;
(2) *For all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x}\rangle + \rho f(y) - \rho f(\hat{x}) \ge 0, \quad \forall y \in C;$$

(3) *If E is strictly convex, then Π_C^f is a single-valued mapping.*

Lemma 2.9 (Li *et al.* [28]) *Let E be a real reflexive smooth Banach space and C be a nonempty closed and convex subset of E. If $\hat{x} \in \Pi_C^f x$ for all $x \in E$, then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \le G(y, Jx), \quad \forall y \in C.$$

Remark 2.10 Let E be a uniformly convex and uniformly smooth Banach space and $f(x) = 0$ for all $x \in E$. Then Lemma 2.9 reduces to the property of the generalized projection operator considered by Alber [14].

If $f(y) \ge 0$ for all $y \in C$ and $f(0) = 0$, then the definition of totally quasi-ϕ-asymptotically nonexpansive T is equivalent to the following:

If $F(T) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$G\bigl(p, JT^n x\bigr) \le G(p, Jx) + \nu_n \zeta G(p, Jx) + \mu_n, \quad \forall x \in C, p \in F(T), n \ge 1.$$

3 Main results

Theorem 3.1 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $\{T_i\}_{i=1}^\infty$ be a countable family of closed and uniformly totally quasi-ϕ-asymptotically nonexpansive mappings with the sequences ν_n, μ_n of nonnegative real numbers with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Let $f : E \to \mathbb{R}$ be a* *convex and lower semicontinuous function with* $C \subset \operatorname{int}(D(f))$ *such that* $f(x) \geq 0$ *for all* $x \in C$ *and* $f(0) = 0$. *Assume that* T_i *is uniformly asymptotically regular for all* $i \geq 1$ *and* $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *For an initial point* $x_1 \in E$, *let* $C_{1,i} = C$ *for each* $i \geq 1$ *and* $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$ *and define the sequence* $\{x_n\}$ *by*

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_i^n x_n), \\ C_{n+1,i} = \{z \in C_n : G(z, Jy_{n,i}) \leq G(z, Jx_n) + \beta_n\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_1, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\beta_n = \nu_n \sup_{q \in \mathcal{F}} \psi G(q, Jx_n) + \mu_n$. *If* $\liminf_{n\to\infty}(1-\alpha_n) > 0$, *then* $\{x_n\}$ *converges strongly to a point* $\Pi_{\mathcal{F}}^f x_1$.

Proof We split the proof into four steps.

Step 1. We first show that C_{n+1} is closed and convex for all $n \geq 1$. From the definition, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i} = C$ for all $i \geq 1$ is closed and convex. Suppose that $C_{n,i}$ is closed and convex for some $n \geq 1$. For any $z \in C_{n,i}$, we know that $G(z, Jy_{n,i}) \leq G(z, Jx_n) + \beta_n$ is equivalent to the following:

$$2\langle z, Jx_n - Jy_{n,i}\rangle \leq \|x_n\|^2 - \|y_{n,i}\|^2 + \beta_n, \quad \forall i \geq 1.$$

Therefore, $C_{n+1,i}$ is closed and convex. Hence $C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}$ is closed and convex for all $n \geq 1$.

Step 2. We show, by induction, that $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \subset C_n$ for all $n \geq 1$. It is obvious that $\mathcal{F} \subset C_1 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \geq 1$. Let $q \in \mathcal{F}$. Since $\{T_i\}$ is a totally quasi-ϕ asymptotically nonexpansive mapping, for each $i \geq 1$, we have

$$\begin{aligned} &G(q, Jy_{n,i}) \\ &\quad = G\big(q, \alpha_n Jx_n + (1-\alpha_n)JT_i^n x_n\big) \\ &\quad = \|q\|^2 - 2\big\langle q, \alpha_n Jx_n + (1-\alpha_n)JT_i^n x_n\big\rangle + \big\|\alpha_n Jx_n + (1-\alpha_n)JT_i^n x_n\big\|^2 + 2\rho f(q) \\ &\quad \leq \|q\|^2 - 2\alpha_n\langle q, Jx_n\rangle - 2(1-\alpha_n)\big\langle q, JT_i^n x_n\big\rangle + \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\big\|JT_i^n x_n\big\|^2 + 2\rho f(q) \\ &\quad = \alpha_n G(q, Jx_n) + (1-\alpha_n)G\big(q, JT_i^n x_n\big) \\ &\quad \leq \alpha_n G(q, Jx_n) + (1-\alpha_n)\big(G(q, Jx_n) + \nu_n\psi\big(G(q, Jx_n)\big) + \mu_n\big) \\ &\quad \leq G(q, Jx_n) + \nu_n \sup \psi\big(G(q, Jx_n)\big) + \mu_n \\ &\quad = G(q, Jx_n) + \beta_n. \end{aligned} \tag{3.2}$$

This shows that $q \in C_{n+1}$, which implies that $\mathcal{F} \subset C_{n+1}$. Hence $\mathcal{F} \subset C_n$ for all $n \geq 1$.

Step 3. We show that $x_n \to p$ as $n \to \infty$. Since $f : E \to \mathbb{R}$ is a convex and lower semicontinuous function, from Lemma 2.7, it follows that there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that $f(x) \geq \langle x, x^*\rangle + \alpha$ for all $x \in E$. Since $x_n \in E$, it follows that

$$\begin{aligned} G(x_n, Jx_1) &= \|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2 + 2\rho f(x_n) \\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2 + 2\rho\big\langle x_n, x^*\big\rangle + 2\rho\alpha \end{aligned}$$

$$\begin{aligned} &= \|x_n\|^2 - 2\langle x_n, Jx_1 - \rho x^*\rangle + \|x_1\|^2 + 2\rho\alpha \\ &\ge \|x_n\|^2 - 2\|x_n\|\,\|Jx_1 - \rho x^*\| + \|x_1\|^2 + 2\rho\alpha \\ &= \left(\|x_n\| - \|Jx_1 - \rho x^*\|\right)^2 + \|x_1\|^2 - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha. \end{aligned} \tag{3.3}$$

For any $q \in \mathcal{F}$, since $x_n = \Pi^f_{C_n} x_1$, we have

$$G(q, Jx_1) \ge G(x_n, Jx_1) \ge \left(\|x_n\| - \|Jx_1 - \rho x^*\|\right)^2 + \|x_1\|^2 - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha.$$

This implies that $\{x_n\}$ is bounded and so are $\{G(x_n, Jx_1)\}$ and $\{y_{n,i}\}$. From the fact that $x_{n+1} = \Pi^f_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ and $x_n = \Pi^f_{C_n} x_1$, it follows from Lemma 2.9 that

$$0 \le \left(\|x_{n+1} - x_n\|\right)^2 \le \phi(x_{n+1}, x_n) \le G(x_{n+1}, Jx_1) - G(x_n, Jx_1). \tag{3.4}$$

This implies that $\{G(x_n, Jx_1)\}$ is nondecreasing. Hence we know that $\lim_{n\to\infty} G(x_n, Jx_1)$ exists. Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.5}$$

Since $\{x_n\}$ is bounded, E is reflexive and C_n is closed and convex for all $n \ge 1$, we can assume that $x_n \rightharpoonup p \in C_n$. From the fact that $x_n = \Pi^f_{C_n} x_1$ and $p \in C_n$, we get

$$G(x_n, Jx_1) \le G(p, Jx_1), \quad \forall n \ge 1. \tag{3.6}$$

Since f is convex and lower semicontinuous, we have

$$\begin{aligned} \liminf_{n\to\infty} G(x_n, Jx_1) &= \liminf_{n\to\infty} \left\{\|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2 + 2\rho f(x_n)\right\} \\ &\ge \|p\|^2 - 2\langle p, Jx_1\rangle + \|x_1\|^2 + 2\rho f(p) \\ &= G(x_n, Jx_1). \end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we get

$$G(p, Jx_1) \le \liminf_{n\to\infty} G(x_n, Jx_1) \le \limsup_{n\to\infty} G(x_n, Jx_1) \le G(p, Jx_1).$$

That is, $\lim_{n\to\infty} G(x_n, Jx_1) = G(p, x_1)$, which implies that $\|x_n\| \to \|p\|$ and so, by virtue of the Kadec-Klee property of E, it follows that

$$\lim_{n\to\infty} x_n = p. \tag{3.8}$$

We also have

$$\lim_{n\to\infty} x_{n+1} = p. \tag{3.9}$$

Since $\{x_n\}$ is bounded (we denote $M = \sup_{n\ge 0}\{\|x_n\|\} < \infty$), it follows that

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \left(\nu_n \sup_{q\in\mathcal{F}} \psi\left(G(q, x_n)\right) + \mu_n\right) = 0. \tag{3.10}$$

From (3.8) and (3.9), we have $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. Since J is uniformly norm-to-norm continuous, it follows that

$$\lim_{n\to\infty}\|Jx_n - Jx_{n+1}\| = 0. \tag{3.11}$$

Since $x_{n+1} = \Pi^f_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$ and by the definition of C_{n+1}, it follows that

$$\begin{aligned}
&G(x_{n+1}, Jy_{n,i}) \le G(x_{n+1}, Jx_n) + \beta_n, \quad \forall i \ge 1 \\
&\iff \quad \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_{n,i}\rangle + \|y_{n,i}\|^2 + 2\rho f(x_{n+1}) \\
&\qquad\qquad \le \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_n\rangle + \|x_n\|^2 + 2\rho f(x_{n+1}) + \beta_n, \quad \forall i \ge 1 \\
&\iff \quad \phi(x_{n+1}, y_{n,i}) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_{n,i}\rangle + \|y_{n,i}\|^2 \\
&\qquad\qquad\qquad\qquad \le \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_n\rangle + \|x_n\|^2 + \beta_n \\
&\qquad\qquad\qquad\qquad = \phi(x_{n+1}, x_n) + \beta_n, \quad \forall i \ge 1,
\end{aligned}$$

that is, we get

$$\phi(x_{n+1}, y_{n,i}) \le \phi(x_{n+1}, x_n) + \beta_n, \quad \forall i \ge 1.$$

From (3.5) and (3.10), it follows that for each $i \ge 1$,

$$\lim_{n\to\infty}\phi(x_{n+1}, y_{n,i}) = 0. \tag{3.12}$$

Also, from (1.2), it follows that for each $i \ge 1$,

$$\|y_{n,i}\| \to \|p\| \quad (n \to \infty). \tag{3.13}$$

Since J is uniformly norm-to-norm continuous, it follows that for each $i \ge 1$,

$$\|Jy_{n,i}\| \to \|Jp\| \quad (n \to \infty). \tag{3.14}$$

That is, $\{\|Jy_{n,i}\|\}$ bounded in E^* for all $i \ge 1$. Since E is reflexive and E^* is also reflexive, we can assume that $Jy_{n,i} \rightharpoonup y^* \in E^*$ for all $i \ge 1$. Since E is reflexive, we see that $J(E) = E^*$. Hence there exists $y \in E$ such that $Jy = y^*$. It follows that for each $i \ge 1$,

$$\phi(x_{n+1}, y_{n,i}) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_{n,i}\rangle + \|y_{n,i}\|^2 = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_{n,i}\rangle + \|Jy_{n,i}\|^2.$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above and the property of weak lower semicontinuity of the norm $\|\cdot\|$, it follows that

$$\begin{aligned}
0 &\ge \|p\|^2 - 2\langle p, y^*\rangle + \|y^*\|^2 \\
&= \|p\|^2 - 2\langle p, Jy\rangle + \|Jy\|^2 \\
&= \|p\|^2 - 2\langle p, Jy\rangle + \|y\|^2 \\
&= \phi(p, y).
\end{aligned}$$

That is, $p = y$, which implies that $y^* = Jp$. It follows that for each $i \geq 1$, $Jy_{n,i} \rightharpoonup Jp \in E^*$. From (3.14) and the Kadec-Klee property of E^*, we have $Jy_{n,i} \to Jp$ as $n \to \infty$ for all $i \geq 1$. Since $J^{-1} : E^* \to E$ is norm-weak*-continuous, that is, $y_{n,i} \rightharpoonup p$, it follows from (3.13) and the Kadec-Klee property of E that

$$\lim_{n\to\infty} y_{n,i} = p, \quad \forall i \geq 1. \tag{3.15}$$

From (3.9), (3.15) and the triangle inequality, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_{n,i}\| = 0, \quad \forall i \geq 1. \tag{3.16}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_{n,i}\| = 0, \quad \forall i \geq 1. \tag{3.17}$$

From the definition of $y_{n,i}$, it follows that

$$\begin{aligned} \|Jx_{n+1} - Jy_{n,i}\| &= \left\|Jx_{n+1} - \alpha_n Jx_n - (1-\alpha_n)JT_i^n x_n\right\| \\ &= \left\|(1-\alpha_n)Jx_{n+1} - (1-\alpha_n)JT_i^n x_n + \alpha_n Jx_{n+1} - \alpha_n Jx_n\right\| \\ &\geq (1-\alpha_n)\left\|Jx_{n+1} - JT_i^n x_n\right\| - \alpha_n \|Jx_n - Jx_{n+1}\|, \end{aligned} \tag{3.18}$$

and so

$$\left\|Jx_{n+1} - JT_i^n x_n\right\| \leq \frac{1}{(1-\alpha_n)}\left(\|Jx_{n+1} - Jy_{n,i}\| + \alpha_n \|Jx_n - Jx_{n+1}\|\right). \tag{3.19}$$

Since $\liminf_{n\to\infty}(1-\alpha_n) > 0$, it follows from (3.11) and (3.17) that, for each $i \geq 1$,

$$\lim_{n\to\infty} \left\|Jx_{n+1} - JT_i^n x_n\right\| = 0. \tag{3.20}$$

Since J^{-1} is uniformly norm-to-norm continuous, for each $i \geq 1$, we obtain

$$\lim_{n\to\infty} \left\|x_{n+1} - T_i^n x_n\right\| = 0. \tag{3.21}$$

By using the triangle inequality, for each $i \geq 1$, we have

$$\left\|T_i^n x_n - p\right\| \leq \left\|T_i^n x_n - x_{n+1}\right\| + \|x_{n+1} - p\|.$$

From (3.21) and $x_{n+1} \to p$ as $n \to \infty$, it follows that for each $i \geq 1$,

$$\lim_{n\to\infty} \left\|T_i^n x_n - p\right\| = 0. \tag{3.22}$$

For each $i \geq 1$, we have

$$\left\|T_i^{n+1} x_n - p\right\| \leq \left\|T_i^{n+1} x_n - T_i^n x_n\right\| + \left\|T_i^n x_n - p\right\|.$$

Since T_i is uniformly asymptotically regular for all $i \geq 1$, it follows from (3.22) that

$$\left\| T_i^{n+1} x_n - p \right\| = 0. \tag{3.23}$$

That is, $T_i^{n+1} x_n = T_i T_i^n x_n \to p$ as $n \to \infty$. From $T_i^n x_n \to p$ as $n \to \infty$ and the closedness of T_i, we have $T_i p = p$ for all $i \geq 1$. We see that $p \in F(T_i)$ for all $i \geq 1$, which implies that $p \in \bigcap_{i=1}^{\infty} F(T_i)$.

Step 4. We show that $p = \Pi_{\mathcal{F}}^f x_1$. Since $\mathcal{F}$ is a closed and convex set, it follows from Lemma 2.8 that $\Pi_{\mathcal{F}}^f x_1$ is single-valued, which is denoted by v. By the definition $x_n = \Pi_{C_n}^f x_1$ and $v \in \mathcal{F} \subset C_n$, we also have

$$G(x_n, Jx_1) \leq G(v, Jx_1), \quad \forall n \geq 1.$$

By the definition of G and f, we know that, for any $x \in E$, $G(\xi, Jx)$ is convex and lower semicontinuous with respect to ξ and so

$$G(p, Jx_1) \leq \liminf_{n\to\infty} G(x_n, Jx_1) \leq \limsup_{n\to\infty} G(x_n, Jx_1) \leq G(v, Jx_1).$$

From the definition of $\Pi_{\mathcal{F}}^f x_1$, since $p \in \mathcal{F}$, we conclude that $v = p = \Pi_{\mathcal{F}}^f x_1$ and $x_n \to p$ as $n \to \infty$. This completes the proof. □

Setting $\nu_n \equiv 0$ and $\mu_n \equiv 0$ in Theorem 3.1, we have the following.

Corollary 3.2 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings. Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset \operatorname{int}(D(f))$ such that $f(x) \geq 0$ for all $x \in C$ and $f(0) = 0$. Assume that T_i is uniformly asymptotically regular for all $i \geq 1$ and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. For an initial point $x_1 \in E$, let $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT_i^n x_n), \\ C_{n+1,i} = \{z \in C_n : G(z, Jy_{n,i}) \leq G(z, Jx_n)\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_1, \quad \forall n \geq 1, \end{cases} \tag{3.24}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}}^f x_1$.

Setting $i = 1$ and $T_i = T$ in Theorem 3.1, we have the following.

Corollary 3.3 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let T be a closed totally quasi-ϕ-asymptotically nonexpansive mapping with the sequences ν_n, μ_n of nonnegative real numbers with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous* *function with $C \subset \operatorname{int}(D(f))$ such that $f(x) \geq 0$ for all $x \in C$ and $f(0) = 0$. Assume that T is a uniformly asymptotically regular and $\mathcal{F} = F(T) \neq \emptyset$. For an initial point $x_1 \in E$, let $C_1 = C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ C_{n+1} = \{z \in C_n : G(z, Jy_{n,i}) \leq G(z, Jx_n) + \beta_n\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.25}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\beta_n = \nu_n \sup \psi G(p, x_n) + \mu_n$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to a point $\Pi^f_{\mathcal{F}} x_1$.

Taking $f(x) = 0$ for all $x \in C$, we have $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi^f_C x = \Pi_C x$. Thus, from Theorem 3.1, we obtain the following.

Corollary 3.4 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of closed and uniformly totally quasi-ϕ-asymptotically nonexpansive mappings with the sequences ν_n, μ_n of nonnegative real numbers with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that T_i is uniformly asymptotically regular for all $i \geq 1$ and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. For an initial point $x_1 \in E$, let $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_i^n x_n), \\ C_{n+1,i} = \{z \in C_n : \phi(z, y_{n,i}) \leq \phi(z, x_n) + \beta_n\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.26}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\beta_n = \nu_n \sup \psi(\phi(p, x_n)) + \mu_n$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_1$.

Corollary 3.5 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings. Assume that T_i is uniformly asymptotically regular for all $i \geq 1$ and $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. For an initial point $x_1 \in E$, let $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_i^n x_n), \\ C_{n+1,i} = \{z \in C_n : \phi(z, y_{n,i}) \leq \phi(z, x_n)\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.27}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_1$.

Corollary 3.6 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let T be a closed totally* *quasi-ϕ-asymptotically nonexpansive mapping with the sequences ν_n, μ_n of nonnegative real numbers with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that T is uniformly asymptotically regular and $\mathcal{F} = F(T) \neq \emptyset$. For an initial point $x_1 \in E$, let $C_1 = C$ and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \beta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.28}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\beta_n = \nu_n \sup \psi(\phi(p, x_n)) + \mu_n$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_1$.

Remark 3.7 (1) Corollary 3.3 extends and improves the results of Li *et al.* [28] from a relatively nonexpansive mapping to a totally quasi-ϕ-asymptotically nonexpansive mapping.

(2) Corollary 3.6 extends and generalizes the result of Takahashi *et al.* [21] from a Hilbert space to a Banach space and from a nonexpansive mapping to a totally quasi-ϕ asymptotically nonexpansive mapping.

(3) In the case of spaces, we extend Banach spaces from a uniformly smooth and uniformly convex Banach to a uniformly smooth and strictly convex Banach with the Kadec-Klee property, which can be found in the literature works by many authors (see [12, 21, 22, 28]).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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